A Remarkable eta-product Identity

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(draft version 7)

Early in 2005 I discovered a remarkable eta-product identity that was known most probably to a very few people implicitly and even fewer explicitly. In order to state the identity we first define the infinite product

$$h(q) := q^{(1/24)}(1-q)(1-q^2)(1-q^3)(1-q^4)...$$

where $|q| < 1$ is required for convergence. This is the Dedekind eta function defined in terms of q. For brevity let $e_n := h(q^n)$. For example, $e_1 = h(q), e_2 = h(q^2)$, and so on. Then for all $|q| < 1$, the following three term identity holds

$$e_2e_6e_{10}e_{30} = e_1e_{12}e_{15}e_{20} + e_3e_4e_5e_{60}.$$

Notice that each term is the product of distinct eta-function factors. That is, each term is linear in each factor of $h(q^n)$ which appears. This seems to be the only identity of its kind. More precisely, here is my conjecture :

Given a finite set S of positive integers, define the eta-product $E(S)$ to be the product over all n in S of $h(q^n)$. Then there exists a unique triple (S_1, S_2, S_3) such that $E(S_1) = E(S_2) + E(S_3)$ where the three sets are pairwise disjoint and the GCD of the union is 1. That triple is $(\{2, 6, 10, 30\}, \{1, 12, 15, 20\}, \{3, 4, 5, 60\})$. Of course, S_2 and S_3 can be swapped giving the same identity without affecting uniqueness.

I conjecture this based on my database of thousands of eta-product identities which I update frequently and is available upon request. As evidence I have used systematic search by computer program up to level 180 and have found no other identity like this one. However, a close contender is the three term identity triple $(\{2, 7, 9, 28, 36, 126\}, \{1, 4, 14, 18, 63, 252\}, \{2, 6, 14, 18, 42, 126\})$ but the sets are not pairwise disjoint. Another close contender is the four term identity with one term having a numerical coefficient of 2

$$e_1e_3e_5e_{15} + 2e_2e_6e_{10}e_{30} = e_1e_2e_{15}e_{30} + e_3e_5e_6e_{10}.$$

My unique three term identity is related to the McKay-Thompson series of class 60D for which see sequences A058728 and A143751 in Neil Sloane's OEIS and more details of which are in my frequently updated database on Monstrous Moonshine also available upon request. More information about $h(q)$ is in my essay "A Multisection of q-Series" which is available on the web.